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Some studies on strongly π-regular rings

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SOME STUDIES ON STRONGLY π-REGULAR RINGS

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In his paper [10], H. Tominaga proved that if R is a π-regular ring of bounded index then $(R)_n$ is strongly π-regular for any positive integer n. The present paper is motivated by this result. In Theorem 1, we shall prove the same for N-rings. Moreover, we shall give other equivalent conditions for a N-ring to be strongly π-regular. In particular, our theorem is true for one-sided duo rings and CN-rings. We attempt also similar investigations for CI-rings, left s-unital Noetherian rings and PI-rings (Theorems 2, 3 and 4). Finally, in Theorem 5 we shall show that any PI-ring contains a unique maximal π-regular ideal.

Throughout R will represent a ring (possibly without 1), N_0 the set of all nilpotent elements of R, N the prime radical of R, and J the Jacobson radical of R.

1. The present section is devoted to exhibit several preliminary results. An element a in R is called a *semi-unit* if there exists an element a' (called a *semi-inverse* of a) such that $a^2a' = a, a'^2a = a'$ and $aa' = a'a$. As was shown in [1], a semi-inverse a' of a is uniquely determined and $a'c = ca'$ for each $c \in R$ with $ac = ca$. First, we summarize the results of [1] in the next

Proposition 1. (1) *Every strongly regular element of R is a semi-unit. In case R is of bounded index n, if a is a right (or left) π-regular element of R then a^n is a semi-unit.*

(2) *If R is a π-regular ring of bounded index then R is strongly π-regular.*

Obviously, [8, Proposition 1. 2] and [8, Théorème 1. 3] are direct consequences of Proposition 1. Although the latter part of Proposition 1 (1) is not true for general rings, F. Dischinger [3] has proved the following

Proposition 2. *Every right (or left) π-regular ring R is strongly π-regular.*

Proof. For the sake of completeness, we shall give here a somewhat economical proof. Given an element x in R, there exist y, $z \in R$ such that $x^{h+1}y = x^h$ and $y^{k+1}z = y^k$ for some positive integers h, k. Setting

$a = x^{h+k}$, $b = y^{h+k}$ and $c = z^{h+k}$, we have $a^2b = a$, $b^2c = b$, and there exists $d \in R$ such that $(c-a)^n = (c-a)^{n+1}d$ for some positive integer n. Evidently, $abc = a^2b^2c = a^2b = a$, whence it follows $ac = a^2bc = a^2$, namely $a(c-a) = 0$. Accordingly, $b^2(c-a)^2 = b^2c(c-a) - b^2a(c-a) = b(c-a)$, and in general

$$(*) \qquad b^m(c-a)^m = b(c-a) \qquad (m = 1, 2 \cdots).$$

Now, noting that $abc = a$, $(c-a)^n = (c-a)^{n+2}d^2$ and $a(c-a) = 0$, the repeated use of (*) enables us to see that $a - ab^2a^2 = ab(c-a) + ab^2(c-a)a = ab^{n+1}(c-a)^{n-1} + ab^{n+1}(c-a)^n a = ab^{n+1}(c-a)^n a = ab^{n+1}(c-a)^{n+2}d^2c = ab(c-a)^2d^2c = (a - aba)(c-a)d^2c = 0$, which proves evidently the left π-regularity of x.

The next is only a restatement of [5, Theorem 2. 1].

Proposition 3. *The following are equivalent*:
1) *R is strongly π-regular.*
2) *R/N is strongly π-regular.*
3) *Every prime factor ring of R is strongly π-regular.*

Proof. It suffices to show that 3) implies 1). Suppose R contains an element a that is not right π-regular. Then, by Zorn's lemma, there exists an ideal I of R that is maximal with respect to the property that $\bar{a}$ is not right π-regular in $\bar{R} = R/I$. Since I can not be prime, there exist ideals K, L properly containing I such that $KL \subseteq I$. Then we can find a positive integer n such that $a^n - a^{2n+1}x \in K$ and $a^n - a^{2n+1}y \in L$ for some $x, y \in R$. But $a^{2n} - a^{2n+1}(a^ny + xa^n - xa^{2n+1}y) = (a^n - a^{2n+1}x)(a^n - a^{2n+1}y) \in KL \subseteq I$, which is a contradiction.

If N coincides with N_0, or equivalently, if R/N is a reduced ring, then R is called an *N-ring*. As was noted in [11], every P_1-ring is an *AC*-ring, and every *AC*-ring is an *N*-ring. Moreover, every right (or left) duo ring is an *N*-ring. In fact, if a is an element of a semi-prime right duo ring R and $a^2 = 0$, then $aRaR \subseteq a(a)R = a|a)R = 0$, whence it follows $aR = 0$ and $a = 0$. By Proposition 1 (1) and Proposition 3, one will easily see that R is a strongly π-regular *N*-ring if and only if R/N is strongly regular. Following [4], R is called a *CN-ring* (resp. *CI-ring*) if every nilpotent (resp. idempotent) element of R is central. As is easily seen, every *CN*-ring is a *CI*-ring (and an *N*-ring), but not conversely. M. P. Drazin [4, Theorem 2] gave the following sufficient condition for R to be a *CN*-ring.

Proposition 4. *If for each $x, y \in R$ there exists some $z \in R$ such that $[x - x^2z, y] = 0$, then R is a CN-ring.*

Proof. Let $x^n = 0$ $(n > 1)$. We shall proceed by the induction with respect to n. Given $y \in R$, there exists z with $[x - x^2z, y] = 0$. Since x^2 is central by $(x^2)^{n-1} = 0$, it follows $(x^2z)^{n-1} = 0$, so that x^2z is central. Hence, $[x, y] = [x^2z, y] = 0$.

In [12], R is called a *right* (resp. *left*) *weakly π-regular ring* if for each $x \in R$ there exists a natural number n such that $a^n \in (a^nR)^2$ (resp. $a^n \in (Ra^n)^2$). Now, R is defined to be *right* (resp. *left*) *π'-regular* if for each $x \in R$ there exists a natural number n such that $x^n = x^nyx^nz$ (resp. $x^n = zx^nyx^n$) with some $y, z \in R$. Needless to say, every π-regular ring is (right and left) π'-regular, and every right (resp. left) π'-regular ring is right (resp. left) weakly π-regular.

2. We shall begin this section with the following

Theorem 1. *If R is an N-ring then the following are equivalent:*

1) *R is strongly π-regular.*
2) *R is π-regular.*
3) *R is right π'-regular*
3') *R is left π'-regular.*
4) *J is nil and R/J is π-regular.*
5) *R/I is π-regular for some nil ideal I.*
6) *R/N is strongly regular.*
7) *Every proper prime ideal of R is a maximal one-sided ideal.*
8) *Every proper completely prime ideal of R is a maximal one-sided ideal.*
9) *$(R)_n$ is strongly π-regular* $(n = 1, 2, \cdots)$.

Proof. Evidently, 2) $\Longrightarrow$ 3) (and 3')). By Proposition 1, 9) $\Longrightarrow$ 1) $\Longrightarrow$ 2) $\Longrightarrow$ 4) $\Longrightarrow$ 5) $\Longrightarrow$ 6). The equivalence of 6) – 8) is immediate by [11, Theorem 8].

3) $\Longrightarrow$ 1) In virtue of Proposition 3, we may restrict our attention to the case that R is reduced. If $x^n = x^nyx^nz$ then $x^nyx^n = x^nyx^nzyx^n$, which implies $(x^nyx^n - zyx^{2n}yx^n)^2 = 0$. Hence, $x^nyx^n = zyx^{2n}yx^n$, and we obtain $x^n = x^nyx^nz = zyx^{2n}yx^nz = zyx^{2n}$. Similarly, 3') $\Longrightarrow$ 1).

6) $\Longrightarrow$ 9) As is well known, the prime radical of $(R)_n$ coincides with $(N)_n$ and $(R)_n/(N)_n$ is isomorphic to $(R/N)_n$. Since $(R/N)_n$ is strongly π-regular by [10, Theorem 5], $(R)_n$ is strongly π-regular by Propositon 3.

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The next includes [2, Theorem 3].

Corollary 1. *If R is a right (or left) duo ring, then the following are equivalent*:

1) *R is strongly π-regular.*
2) *R is π-regular.*
3) *R is right π'-reqular.*
3′) *R is left π'-regular.*
4) *R is right weakly π-regular.*
4′) *R is left weakly π-regular.*
5) *J is nil and R/J is π-regular.*
6) *R/I is π-regular for some nil ideal I.*
7) *R/N is strongly regular.*
8) *Every proper prime ideal of R is maximal.*
9) *Every proper completely prime ideal of R is maximal.*
10) *$(R)_n$ is strongly π-regular* $(n = 1, 2, \cdots)$.

Proof. As was noted in §1, every right duo ring is an N-ring. Moreover, by [12, Corollary 7], every left weakly π-regular, reduced ring is right weakly π-regular. By Proposition 3, it remains therefore to prove that if R is right weakly π-regular and reduced then R is strongly π-regular. Recalling that R is a right duo ring, the last will be easily seen.

A ring R is said to be *restricted π-regular* (resp. *restricted strongly π-regular*) if every proper homomorphic image of R is π-regular (resp. strongly π-regular). The next is a direct consequence of Corollary 1.

Corollary 2. *If R is a right (or left) duo ring, then the following are equivalent*:

1) *R is restricted strongly π-regular.*
2) *R is restricted π-regular.*
3) *Every non-zero proper prime ideal of R is maximal.*
4) *Every non-zero proper completely prime ideal of R is maximal.*

Next, as a combination of Theorem 1 and Proposition 4, we obtain the following that includes [4, Theorem 5] and [8, Théorème 2. 1].

Corollary 3. *The following are equivalent*:

1) *R is a strongly π-regular CN-ring.*
2) *R is a π-regular CN-ring.*
3) *R is a rihgt π'-regular CN-ring.*

3′) *R is a left π^l-regular CN-ring.*

4) *R is a CN-ring whose proper prime ideals are maximal one-sided ideals.*

5) *R is a CN-ring whose proper completely prime ideals are maximal one-sided ideals.*

6) *R is a CN-ring and $(R)_n$ is strongly π-regular ($n = 1, 2, \cdots$).*

7) *For each $x \in R$ there exists some y such that $x - x^2y$ is a central nilpotent element.*

8) *R is a π-regular ring such that for each $x, y \in R$ there exists some z with $[x - x^2z, y] = 0$.*

Now, corresponding to Corollary 3, we shall prove the next

Theorem 2. *If R is a CI ring, then the following are equivalent:*

1) *R is strongly π-regular.*

2) *R is π-regular.*

3) *J is nil and R/J is π-regular.*

4) *R/I is π-regular for some nil ideal I.*

5) *R/N is π-regular.*

6) *Every prime factor ring of R is either a nil ring or a local ring with Jacobson radical nil.*

7) *J is nil and every element of R is either π-regular or quasi-regular.*

8) *Every non-nil right ideal of R contains a non-zero idempotent and every element annihilated by some non-zero idempotent is π-regular.*

Proof. First, we claim that every π-regular element of R is strongly π-regular. In fact, if $x^n = x^nyx^n$ then both x^ny and yx^n are central idempotents. Clearly, 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4), 2) $\Rightarrow$ 5) $\Rightarrow$ 4), and 2) $\Rightarrow$ 7). By Proposition 3, 6) $\Rightarrow$ 1).

4) $\Rightarrow$ 1) Given $x \in R$, there exists $y \in R$ such that $x^nyx^n - x^n \in I$ for some positive integer n. Since $(x^ny)^2 - x^ny \in I$, there exists a (central) idempotent e in the subring $[x^ny]$ generated by x^ny such that $e - x^ny \in I$. We have then $x^ne - x^n \in I$, and hence $(x^ne - x^n)^m = 0$ for some m, whence it follows $x^{nm} = x^{nm}ey'$ with some $y' \in R$. Since e is in $[x^ny]$, this means $x^{nm} = x^{nm+1}y''$ with some $y'' \in R$. Hence, R is right and similarly (or by Proposition 2) left π-regular.

1) $\Rightarrow$ 6) Let R' be a prime homomorphic image of R. As is easily seen, R' is a *CI*-ring. If R' is not nil, then R' contains the identity and every element of R' is either nilpotent or invertible.

7) $\Rightarrow$ 8) Let I be a non-nil right ideal of R. Then there exists an element x in I such that xy is not quasi-regular for some y. Hence,

there exists $z \in R$ such that $(xy)^n = (xy)^n z(xy)^n$. Evidently, $(xy)^n z$ is a non-zero idempotent in I. Next, let e be a non-zero idempotent of R, and $ea = 0$. As is easily seen, $e + a$ is not quasi-regular, so that $(e+a)^m = (e+a)^m b(e+a)^m$ for some $b \in R$ and some positive integer m. Then, one can see $e = ebe$, $0 = eba^m$, $0 = a^m be$, and eventually $a^m = a^m ba^m$.

8) $\Longrightarrow$ 1) Let x be a non-nilpotent element of R. Then xR contains a non-zero (central) idempotent $e = xy$. Since e annihilates $x - xe$, there exists an element z in R such that $(x - xe)^n = (x - xe)^{n+1}z$. Then, $x^n - x^n e = x^{n+1}(z - ez)$, whence it follows $x^n = x^{n+1}(y + z - ez)$. Hence, R is strongly π-regular by Proposition 2.

Remark. If R is a (strongly) π-regular CI-ring, then one can easily see that every homomorphic image of R is also a π-regular CI-ring.

As a combination of Corollary 1 and Theorem 2, we readily obtain

Corollary 4. *If R is commutative, then the following are equivalent*:

1) *R is π-regular.*
2) *J is nil and R/J is π-regular.*
3) *R/I is π-regular for some nil ideal I.*
4) *R/N is regular.*
5) *Every proper prime ideal of R is a maximal ideal.*
6) *J is nil and every element of R is either π-regular or guasi-regular.*
7) *Every non-nil ideal of R contains a non-zero idempotent and every element annihilated by some non-zero idempotent is π-regular.*
8) *$(R)_n$ is strongly π-regular* $(n = 1, 2, \cdots)$.

3. Firstly, we shall prove the next that includes [11, Corollary 3].

Theorem 3. *If R is left s-unital, then the following are equivalent*:

1) *R is a left Artinian ring.*
2) *R is a left Noetherian strongly π-regular ring.*
3) *R is a left Noetherian π-regular ring.*
4) *R is a left Noetherian right π'-regular ring.*

4′) *R is a left Noetherian left π'-regular ring.*

5) *R is a fully left Goldie, strongly π-regular ring.*
6) *R is a fully left Goldie, π-regular ring.*
7) *R is a fully left Goldie, right π'-regular ring.*

7′) *R is a fully left Goldie, left π'-regular ring.*

Proof. By [11, Corollary 3], 1), 2), 3), 5) and 6) are equivalent.

Since 4) (resp. 4′)) implies 7) (resp. 7′)) and 6) implies both 7) and 7′), it remains only to prove that 7) (or 7′)) implies 5). Assume 7) (or 7′)). By [12, Lemma 1 (1)], $\bar{R}=R/N$ is a left Goldie ring with identity. Now, let L be an arbitrary essential left ideal of $\bar{R}$. Then it is well known that L contains a regular element u. Since $\bar{R}$ is von Neumann finite (see, e. g, [6]), it is easy to see that u is a unit, which means $L=\bar{R}$. Hence, $\bar{R}$ is (semi-simple) Artinian, and then R is strongly π-regular by Proposition 3.

From the proof of Theorem 3, one can easily see

Corollary 5. *If R is left s-unital and semi-prime, then the following are equivalent*:

1) *R is an Artinian ring.*
2) *R is a left (or right) Noethrian strongly π-regular ring.*
3) *R is a left (or right) Noetherian π-regular ring.*
4) *R is a left (or right) Noetherian right π'-regular ring.*
4′) *R is a left (or right) Noetherian left π'-regular ring.*
5) *R is a left (or right) Goldie, strongly π-regular ring.*
6) *R is a left (or right) Goldie, π-regular ring.*
7) *R is a left (or right) Goldie, right π'-regular ring.*
7′) *R is a left (or right) Goldie, left π'-regular ring.*

In the rest of this section, our attention will be extensively directed towards *PI*-rings.

Theorem 4. *If R is a PI-ring then the following are equivalent*:

1) *R is strongly π-regular.*
2) *R is π-regular.*
3) *R is right π'-regular.*
3′) *R is left π'-regular.*
4) *R is right weakly π-regular.*
4′) *R is left wealy π-regular.*
5) *$(R)_n$ is strongly π-regular ($n=1, 2, \cdots$).*

Proof. It suffices to prove that 4) $\Longrightarrow$ 1) $\Longrightarrow$ 5).

4) $\Longrightarrow$ 1) Let P be an arbitrary prime ideal of R. Then $\bar{R}=R/P$ has a central quotient ring which is a central simple algebra of finite rank (see [9, Corollary to Theorem 2]). Since the center of $\bar{R}$ is a π-regular domain by [12, Lemma 3], the center of $\bar{R}$ is a field and $\bar{R}$ itself is Artinian (simple). Hence, by Proposition 3, R is strongly π-regular.

1) $\Longrightarrow$ 5) By [7, Theorem 1], we can see that R/N is of bounded index. Hence, $(R/N)_n$ is strongly π-regular by [10, Theorem 5]. Since the prime radical of $(R)_n$ coincides with $(N)_n$ and $(R/N)_n \simeq (R)_n/(N)_n$, $(R)_n$ is strongly π-regular again by Proposition 3.

We shall conclude our study with the following

Theorem 5. *If R is a PI-ring, then R contains a unique maximal π-regular ideal M and R/M has no non-zero π-regular ideals.*

Proof. As is noted in the proof of Theorem 4, $\overline{R} = R/N$ is of bounded index. Hence, there exists a unique maximal π-regular ideal $\overline{M} = M/N$ of $\overline{R}$ ([10, Lemma 4]), and $R/M \simeq \overline{R}/\overline{M}$ has no non-zero π-regular ideals ([10, Theorem 6]). As is well known, the prime radical of the ring M is N. Hence, M is strongly π-regular by Proposition 3, and it is easy to see that M is the unique maximal π-regular ideal of R.

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